

ON THE DE RHAM COMPLEX OF MIXED TWISTOR \mathcal{D} -MODULES

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ABSTRACT. Given a complex manifold S , we introduce for each complex manifold X a t-structure on the bounded derived category of \mathbb{C} -constructible complexes of \mathcal{O}_S -modules on $X \times S$. We prove that the de Rham complex of a holonomic $\mathcal{D}_{X \times S/S}$ -module which is \mathcal{O}_S -flat as well as its dual object is perverse relatively to this t-structure. This result applies to mixed twistor \mathcal{D} -modules.

1. INTRODUCTION

Given a vector bundle V of rank $d \geq 1$ with an integrable connection $\nabla : V \rightarrow \Omega_X^1 \otimes V$ on a complex manifold X of complex dimension n , the sheaf of horizontal sections $V^\nabla = \ker \nabla$ is a locally constant sheaf of d -dimensional \mathbb{C} -vector spaces, and is the only nonzero cohomology sheaf of the de Rham complex $\mathrm{DR}_X(V, \nabla) = (\Omega_X^\bullet \otimes V, \nabla)$. Assume moreover that (V, ∇) is equipped with a harmonic metric in the sense of [19, p. 16]. The twistor construction of [20] produces then a holomorphic bundle \mathcal{V} on the product space $\mathcal{X} = X \times \mathbb{C}$, where the factor \mathbb{C} has coordinate z , together with a holomorphic flat z -connection. By restricting to $\mathcal{X}^* := X \times \mathbb{C}^*$, giving such a z -connection on $\mathcal{V}^* := \mathcal{V}|_{\mathcal{X}^*}$ is equivalent to giving a flat relative connection ∇ with respect to the projection $p : \mathcal{X}^* \rightarrow \mathbb{C}^*$. Similarly, the relative de Rham complex $\mathrm{DR}_{\mathcal{X}^*/\mathbb{C}^*}(\mathcal{V}^*, \nabla)$ has cohomology in degree zero at most, and $(\mathcal{V}^*)^\nabla := \ker \nabla$ is a locally constant sheaf of locally free $p^{-1}\mathcal{O}_{\mathbb{C}^*}$ -modules of rank d .

Holonomic \mathcal{D}_X -modules generalize the notion of a holomorphic bundle with flat connection to objects having (possibly wild) singularities, and a well-known theorem of Kashiwara [2] shows that the solution complex of such a holonomic \mathcal{D}_X -module has \mathbb{C} -constructible cohomology, from which one can deduce that the de Rham complex is of the same kind and more precisely that both are \mathbb{C} -perverse sheaves on X up to a shift by $\dim X$.

The notion of a holonomic \mathcal{D}_X -module with a harmonic metric has been formalized in [14] and [10] under the name of pure twistor \mathcal{D} -module (this

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generalizes holonomic \mathcal{D}_X -modules with regular singularities), and then in [15] and [11] under the name of wild twistor \mathcal{D} -modules (this takes into account arbitrary irregular singularities). More recently, Mochizuki [12] has fully developed the notion of a mixed (possibly wild) twistor \mathcal{D} -module. When restricted to \mathcal{X}^* , such an object contains in its definition two holonomic $\mathcal{D}_{\mathcal{X}^*/\mathbb{C}^*}$ -modules, and we say that both underlie a mixed twistor \mathcal{D} -module

The main result of this article concerns the de Rham complex and the solution complex of such objects.

Theorem 1.1. *The de Rham complex and the solution complex of a $\mathcal{D}_{\mathcal{X}^*/\mathbb{C}^*}$ -module underlying a mixed twistor \mathcal{D} -module are perverse sheaves of $p^{-1}\mathcal{O}_{\mathbb{C}^*}$ -modules (up to a shift by $\dim X$).*

In Section 2, we define the notion of relative constructibility and perversity. This applies to the more general setting where $p : \mathcal{X}^* \rightarrow \mathbb{C}^*$ is replaced by a projection $p_X : \mathcal{X} = X \times S \rightarrow S$, where S is any complex manifold. We usually set $p = p_X$ when X is fixed. On the other hand, we call *holonomic* any coherent $\mathcal{D}_{X \times S/S}$ -module whose relative characteristic variety in $T^*(X \times S/S) = (T^*X) \times S$ is contained in a variety $\Lambda \times S$, where Λ is a conic Lagrangian variety in T^*X . We say that a $\mathcal{D}_{X \times S/S}$ -module is *strict* if it is $p^{-1}\mathcal{O}_S$ -flat.

Theorem 1.2. *The de Rham complex and the solution complex of a strict holonomic $\mathcal{D}_{X \times S/S}$ -module whose dual is also strict are perverse sheaves of $p^{-1}\mathcal{O}_S$ -modules (up to a shift by $\dim X$).*

A $\mathcal{D}_{\mathcal{X}^*/\mathbb{C}^*}$ -module \mathcal{M} underlying a mixed twistor \mathcal{D} -module is strict and holonomic (see [12]). Moreover, Mochizuki has defined a duality functor on the category of mixed twistor \mathcal{D} -modules, proving in particular that the dual of \mathcal{M} as a $\mathcal{D}_{\mathcal{X}^*/\mathbb{C}^*}$ -module is also strict holonomic. Therefore, these results together with Theorem 1.2 imply Theorem 1.1.

Note that, while our definition of perverse objects in the bounded derived category $D^b(p^{-1}\mathcal{O}_S)$ intends to supply a notion of holomorphic family of perverse sheaves, we are not able, in the case of twistor \mathcal{D} -modules, to extend this notion to the case when the parameter $z \in \mathbb{C}^* = S$ also achieves the value zero, and to define a perversity property in the Dolbeault setting of [19] for the associated Higgs module.

2. RELATIVE CONSTRUCTIBILITY IN THE CASE OF A PROJECTION

We keep the setting as above, but X is only assumed to be a real analytic manifold. Given a real analytic map $f : Y \rightarrow X$ between real analytic manifolds, we will denote by f_S (or f if the context is clear) the map $f \times \text{id}_S : Y \times S \rightarrow X \times S$.

2.1. Sheaves of \mathbb{C} -vector spaces and of $p^{-1}\mathcal{O}_S$ -modules. Let $f : Y \rightarrow X$ be such a map. There are functors $f^{-1}, f^!, Rf_*, Rf_!$ between $D^b(\mathbb{C}_{X \times S})$ and $D^b(\mathbb{C}_{Y \times S})$, and functors $f_S^{-1}, f_S^!, Rf_{S,*}, Rf_{S,!}$ between $D^b(p_X^{-1}\mathcal{O}_S)$ and

$D^b(p_Y^{-1}\mathcal{O}_S)$. These functors correspond pairwise through the forgetful functor $D^b(p_X^{-1}\mathcal{O}_S) \rightarrow D^b(\mathbb{C}_{X \times S})$. Indeed, this is clear except for $f_S^!$ and $f^!$. To check it, one decomposes f as a closed immersion and a projection. In the first case, the compatibility follows from the fact that both are equal to $f^{-1}R\Gamma_{f(X)}$ (see [5, Prop. 3.1.12]) and for the case of a projection one uses [5, Prop. 3.1.11 & 3.3.2]. We note also that the Poincaré-Verdier duality theorem [5, Prop. 3.1.10] holds on $D^b(p^{-1}\mathcal{O}_S)$ (see [5, Rem. 3.1.6(i)]). From now on, we will write f^{-1} , etc. instead of f_S^{-1} , etc.

The ring $p_X^{-1}\mathcal{O}_S$ is Noetherian, hence coherent (see [3, Prop. A.14]). For each $s_o \in S$ let us denote by \mathfrak{m}_{s_o} the ideal of sections of \mathcal{O}_S vanishing at s_o and by $i_{s_o}^*$ the functor

$$\begin{aligned} \text{Mod}(p_X^{-1}\mathcal{O}_S) &\longmapsto \text{Mod}(\mathbb{C}_X) \\ F &\longmapsto F \otimes_{p_X^{-1}\mathcal{O}_S} p_X^{-1}(\mathcal{O}_S/\mathfrak{m}_{s_o}). \end{aligned}$$

This functor will be useful for getting properties of $D^b(p_X^{-1}\mathcal{O}_S)$ from well-known properties of $D^b(\mathbb{C}_X)$.

Proposition 2.1. *Let F and F' belong to $D^b(p_X^{-1}\mathcal{O}_S)$. Then, for each $s_o \in S$ there is a well-defined natural morphism*

$$Li_{s_o}^*(R\mathcal{H}om_{p^{-1}(\mathcal{O}_S)}(F, F')) \rightarrow R\mathcal{H}om_{\mathbb{C}_X}(Li_{s_o}^*(F), Li_{s_o}^*(F'))$$

which is an isomorphism in $D^b(\mathbb{C}_X)$.

Proof. Let us fix $s_o \in S$. The existence of the morphism follows from [3, (A.10)]. Moreover, since $p_X^{-1}\mathcal{O}_S$ is a coherent ring as remarked above and $p_X^{-1}(\mathcal{O}_S/\mathfrak{m}_{s_o})$ is $p_X^{-1}\mathcal{O}_S$ -coherent, we can apply the argument given after (A.10) in loc. cit. to show that it is an isomorphism. q.e.d.

Proposition 2.2. *Let F and F' belong to $D^b(p_X^{-1}\mathcal{O}_S)$ and let $\phi : F \rightarrow F'$ be a morphism. Assume the following conditions:*

- (1) *for all $j \in \mathbb{Z}$ and $(x, s) \in X \times S$, $\mathcal{H}^j(F)_{(x,s)}$ and $\mathcal{H}^j(F')_{(x,s)}$ are of finite type over $\mathcal{O}_{S,s}$,*
- (2) *for all $s_o \in S$, the natural morphism*

$$Li_{s_o}^*(\phi) : Li_{s_o}^*(F) \rightarrow Li_{s_o}^*(F')$$

is an isomorphism in $D^b(\mathbb{C}_X)$.

Then ϕ is an isomorphism.

Proof. It is enough to prove that the mapping cone of ϕ is quasi-isomorphic to 0. So we are led to proving that for $F \in D^b(p^{-1}\mathcal{O}_S)$, if $\mathcal{H}^j(F)_{(x,s)}$ are of finite type over $\mathcal{O}_{S,s}$ for all $(x, s) \in X \times S$, and $Li_{s_o}^*(F)$ is quasi-isomorphic to 0 for each $s_o \in S$, then F is quasi-isomorphic to 0.

We may assume that S is an open subset of \mathbb{C}^n with coordinates s^1, \dots, s^n and we will argue by induction on n . Assume $n = 1$. For such an F , for each $s_o \in S$ and any $j \in \mathbb{Z}$ the morphism $(s^1 - s_o^1) : \mathcal{H}^j(F) \rightarrow \mathcal{H}^j(F)$ is an isomorphism, hence $\mathcal{H}^j(F)/(s^1 - s_o^1)\mathcal{H}^j(F) = 0$ and by Nakayama's Lemma, for any $x \in X$, $\mathcal{H}^j(F)_{(x,s_o^1)} = 0$ and the result follows. For $n \geq 2$,

the cone F' of the morphism $(s^n - s_o^n) : F \rightarrow F$ also satisfies $Li_{s_o}^* F' = 0$ for any $s_o' = (s_o^1, \dots, s_o^{n-1})$, hence is zero by induction, so we can argue as in the case $n = 1$. q.e.d.

2.2. S -locally constant sheaves. We say that a sheaf F of \mathbb{C} -vector spaces (resp. $p_X^{-1}\mathcal{O}_S$ -modules) on $X \times S$ is *S -locally constant* if, for each point $(x, s) \in X \times S$, there exists a neighbourhood $U = V_x \times T_s$ of (x, s) and a sheaf $G^{(x,s)}$ of \mathbb{C} -vector spaces (resp. \mathcal{O}_S -modules) on T_s , such that $F|_U \simeq p_U^{-1}G^{(x,s)}$. The category of S -locally constant sheaves is an abelian full subcategory of that of sheaves of $\mathbb{C}_{X \times S}$ -vector spaces (resp. $p^{-1}\mathcal{O}_S$ -modules), which is stable by extensions in the respective categories, by $\mathcal{H}om$ and tensor products. Moreover, if $\pi : Y \times X \times S \rightarrow Y \times S$ is the projection, with X contractible, then, if F' is S -locally constant on $Y \times X \times S$,

- $\pi_* F'$ is S -locally constant on $Y \times S$,
- $R^k \pi_* F' = 0$ if $k > 0$,
- $F' \simeq \pi^{-1} \pi_* F'$.

Applying this to $Y = \{\text{pt}\}$, we find that, if F is S -locally constant, then for each $x \in X$ there exists a connected neighbourhood V_x of x and a \mathbb{C}_S -module (resp. \mathcal{O}_S -module) $G^{(x)}$ such that $F = p_{V_x}^{-1}G^{(x)}$, and one has $G^{(x)} = p_{V_x,*}F|_{V_x \times S} = F|_{\{x\} \times S}$. We shall also denote by $D_{\text{lc}}^b(p_X^{-1}\mathbb{C}_S)$ (resp. $D_{\text{lc}}^b(p_X^{-1}\mathcal{O}_S)$) the bounded triangulated category whose objects are the complexes having S -locally constant cohomology sheaves. Similarly, for such a complex F we have $F|_{V_x \times S} \simeq p_{V_x}^{-1}Rp_{V_x,*}F|_{V_x \times S} \simeq p_{V_x}^{-1}F|_{\{x\} \times S}$.

We conclude from the previous remarks, by using the natural forgetful functor $D^b(p_X^{-1}\mathcal{O}_S) \rightarrow D^b(\mathbb{C}_{X \times S})$:

Lemma 2.3.

- (1) An object F of $D^b(p_X^{-1}\mathcal{O}_S)$ belongs to $D_{\text{lc}}^b(p_X^{-1}\mathcal{O}_S)$ if and only if, when regarded as an object of $D^b(\mathbb{C}_{X \times S})$, it belongs to $D_{\text{lc}}^b(p_X^{-1}\mathbb{C}_S)$.
- (2) For any object F of $D_{\text{lc}}^b(p_X^{-1}\mathcal{O}_S)$ and for any $s_o \in S$, $Li_{s_o}^* F$ belongs to $D_{\text{lc}}^b(\mathbb{C}_X)$.

2.3. S -weakly \mathbb{R} -constructible sheaves. As long as the manifold X is fixed, we shall write p instead of p_X .

Definition 2.4. Let $F \in D^b(\mathbb{C}_{X \times S})$ (resp. $F \in D^b(p^{-1}\mathcal{O}_S)$). We shall say that F is *S -weakly \mathbb{R} -constructible* if there exists a subanalytic μ -stratification (X_α) of X (see [5, Def. 8.3.19]) such that, for all $j \in \mathbb{Z}$, $\mathcal{H}^j(F)|_{X_\alpha \times S}$ is S -locally constant.

This condition is independent of the choice of the μ -stratification and characterizes a full triangulated subcategory $D_{\text{w-}\mathbb{R}\text{-c}}^b(p^{-1}\mathbb{C}_S)$ (resp. $D_{\text{w-}\mathbb{R}\text{-c}}^b(p^{-1}\mathcal{O}_S)$) of $D^b(\mathbb{C}_{X \times S})$ (resp. $D^b(p^{-1}\mathcal{O}_S)$). Due to Lemma 2.3, an object F of $D^b(p^{-1}\mathcal{O}_S)$ is in $D_{\text{w-}\mathbb{R}\text{-c}}^b(p^{-1}\mathcal{O}_S)$ if and only if it belongs to $D_{\text{w-}\mathbb{R}\text{-c}}^b(p^{-1}\mathbb{C}_S)$ when considered as an object of $D^b(\mathbb{C}_{X \times S})$. By mimicking for $D_{\text{w-}\mathbb{R}\text{-c}}^b(p^{-1}\mathbb{C}_S)$ the proof of [5, Prop. 8.4.1] and according to the previous remark for $D_{\text{w-}\mathbb{R}\text{-c}}^b(p^{-1}\mathcal{O}_S)$, we obtain:

Proposition 2.5. *Let F be S -weakly \mathbb{R} -constructible on X and let $X = \bigsqcup_{\alpha} X_{\alpha}$ be a μ -stratification of X adapted to F . Then the following conditions are equivalent:*

- (1) *for all $j \in \mathbb{Z}$ and for all α , $\mathcal{H}^j(F)|_{X_{\alpha} \times S}$ is S -locally constant.*
- (2) *$SS(F) \subset (\bigsqcup_{\alpha} T_{X_{\alpha}}^* X) \times T^*S$.*
- (3) *There exists a closed conic subanalytic Lagrangian subset Λ of T^*X such that $SS(F) \subset \Lambda \times T^*S$.*

Proposition 2.6. *Let $F \in D_{w-\mathbb{R}-c}^b(p_X^{-1}\mathcal{O}_S)$ and let $s_o \in S$. Then $Li_{s_o}^*(F) \in D_{w-\mathbb{R}-c}^b(\mathbb{C}_X)$.*

Proof. Let $i_{\alpha} : X_{\alpha} \hookrightarrow X$ denote the locally closed inclusion of a stratum of an adapted stratification (X_{α}) . It is enough to observe that, for each α , we have $i_{\alpha}^{-1}Li_{s_o}^*(F) \simeq Li_{s_o}^*(i_{\alpha}^{-1}F)$, and to apply Lemma 2.3(2). q.e.d.

Let now Y be another real analytic manifold and consider a real analytic map $f : Y \rightarrow X$. The following statements for objects of $D_{w-\mathbb{R}-c}^b(p^{-1}\mathbb{C}_S)$ are easily deduced from Proposition 2.5 similarly to the absolute case treated in [5], as consequences of Theorem 8.3.17, Proposition 8.3.11, Corollary 6.4.4 and Proposition 5.4.4 of loc.cit. In order to get the same statements for objects of $D_{w-\mathbb{R}-c}^b(p^{-1}\mathcal{O}_S)$, one uses Lemma 2.3(1) together with §2.1. We will not distinguish between f and f_S .

Proposition 2.7.

- (1) *If F is S -weakly \mathbb{R} -constructible on X , then so are $f^{-1}(F)$ and $f^!(F)$.*
- (2) *Assume that F' is S -weakly \mathbb{R} -constructible on Y and that f is proper on $\text{Supp}(F')$. Then $Rf_*(F')$ is S -weakly \mathbb{R} -constructible on X .*

Given a closed subanalytic subset $Y \subset X$, we will denote by $i : Y \times S \hookrightarrow X \times S$ the closed inclusion and by j the complementary open inclusion.

Corollary 2.8. *Assume that F^* is S -weakly \mathbb{R} -constructible on $X \setminus Y$. Then the objects $Rj_!F^*$ and Rj_*F^* are also S -weakly \mathbb{R} -constructible on X .*

Proof. The statement for $Rj_!F^*$ is obvious. Then Proposition 2.7 implies that $i^!Rj_!F^*$ is S -weakly \mathbb{R} -constructible. Conclude by using the distinguished triangle

$$Ri_*i^!Rj_!F^* \rightarrow Rj_!F^* \rightarrow Rj_*F^* \xrightarrow{+1}$$

and the S -weak \mathbb{R} -constructibility of the first two terms. q.e.d.

Proposition 2.9. *An object $F \in D^b(\mathbb{C}_{X \times S})$ (resp. $F \in D^b(p^{-1}(\mathcal{O}_S))$) is S -weakly \mathbb{R} -constructible with respect to a μ -stratification (X_{α}) if and only if, for each α , $i_{\alpha}^!F$ has S -locally constant cohomology on X_{α} .*

Proof. Assume that F is S -weakly \mathbb{R} -constructible with respect to a μ -stratification (X_{α}) of X . Then $i_{\alpha}^!F$ has S -locally constant cohomology on X_{α} . Indeed the estimation of the micro-support of [5, Cor. 6.4.4(ii)] implies that $SS(i_{\alpha}^!F)$ (like $SS(i_{\alpha}^*F)$) is contained in $T_{X_{\alpha}}^*X_{\alpha} \times T^*S$, so $i_{\alpha}^!F$ has locally constant cohomology on X_{α} for each α , according to Proposition 2.5.

Conversely, if $i_\alpha^! F$ is locally constant for each α , then F is S -weakly \mathbb{R} -constructible. Indeed, we argue by induction and we denote by X_k the union of strata of codimension $\leq k$ in X . Assume we have proved that $F|_{X_{k-1} \times S}$ is S -weakly \mathbb{R} -constructible with respect to the stratification (X_α) with $\text{codim } X_\alpha \leq k-1$. We denote by $j_k : X_{k-1} \hookrightarrow X_k$ the open inclusion and by i_k the complementary closed inclusion. According to Corollary 2.8, $Rj_{k,*} j_k^{-1} F$ is S -weakly \mathbb{R} -constructible with respect to $(X_\alpha)|_{X_k}$. Now, by using the exact triangle $i_k^! F \rightarrow i_k^{-1} F \rightarrow i_k^{-1} Rj_{k,*} j_k^{-1} F \xrightarrow{+1}$, we conclude that $i_k^{-1} F$ is locally constant, hence $F|_{X_k \times S}$ is S -weakly \mathbb{R} -constructible. q.e.d.

Corollary 2.10. *Let $F, F' \in D_{w\text{-}\mathbb{R}\text{-c}}^b(p_X^{-1} \mathcal{O}_S)$. Then $R\mathcal{H}om_{p_X^{-1} \mathcal{O}_S}(F, F')$ also belongs to $D_{w\text{-}\mathbb{R}\text{-c}}^b(p_X^{-1} \mathcal{O}_S)$.*

Proof. In view of Proposition 2.9, it is sufficient to prove that for each α , $i_\alpha^! R\mathcal{H}om_{p_X^{-1} \mathcal{O}_S}(F, F')$ belongs to $D_{lc}^b(p_X^{-1} \mathcal{O}_S)$. We have:

$$i_\alpha^! R\mathcal{H}om_{p_X^{-1} \mathcal{O}_S}(F, F') \simeq R\mathcal{H}om_{p_\alpha^{-1} \mathcal{O}_S}(i_\alpha^{-1} F, i_\alpha^! F').$$

Since both $i_\alpha^{-1} F$ and $i_\alpha^! F'$ belong to $D_{lc}^b(p_X^{-1} \mathcal{O}_S)$, according to Proposition 2.9, we have locally on X_α isomorphisms $i_\alpha^{-1} F = p_\alpha^{-1} G_\alpha$ and $i_\alpha^! F' = p_\alpha^{-1} G'_\alpha = p_\alpha^! G'_\alpha[-\dim_{\mathbb{R}} X_\alpha]$ for some \mathcal{O}_S -modules G_α and G'_α . Then

$$\begin{aligned} R\mathcal{H}om_{p_\alpha^{-1} \mathcal{O}_S}(i_\alpha^{-1} F, i_\alpha^! F') &= R\mathcal{H}om_{p_\alpha^{-1} \mathcal{O}_S}(p_\alpha^{-1} G_\alpha, p_\alpha^! G'_\alpha[-\dim_{\mathbb{R}} X_\alpha]) \\ &\simeq p_\alpha^! R\mathcal{H}om_{\mathcal{O}_S}(G_\alpha, G'_\alpha)[- \dim_{\mathbb{R}} X_\alpha] \\ &= p_\alpha^{-1} R\mathcal{H}om_{\mathcal{O}_S}(G_\alpha, G'_\alpha). \end{aligned} \quad \text{q.e.d.}$$

The following lemma will be useful in the next section. Assume that $X = Y \times Z$ and that the μ -stratification (X_α) of X takes the form $X_\alpha = Y \times Z_\alpha$, where (Z_α) is a μ -stratification of Z . We denote by $q : X \rightarrow Y$ the projection. Let $z_o \in Z$, let $U \ni z_o$ be a coordinate neighbourhood of z_o in Z and, for each $\varepsilon > 0$ small enough, let $B_\varepsilon \subset U$ be the open ball of radius ε centered at z_o and let \overline{B}_ε be the closed ball and S_ε its boundary. For the sake of simplicity, we denote by $q_\varepsilon, q_{\overline{\varepsilon}}, q_{\partial\varepsilon}$ the corresponding projections.

We set $Z^* = Z \setminus \{z_o\}$ and $X^* = Y \times Z^*$. We denote by $i : Y \times \{z_o\} \hookrightarrow Y \times Z$ and by $j : Y \times Z^* \hookrightarrow Y \times Z$ the complementary closed and open inclusions.

Lemma 2.11. *Let $F^* \in D_{w\text{-}\mathbb{R}\text{-c}}^b(p_{X^*}^{-1} \mathbb{C}_S)$ (resp. $F^* \in D_{w\text{-}\mathbb{R}\text{-c}}^b(p_{X^*}^{-1} \mathcal{O}_S)$) be adapted to the previous stratification. Then there exists $\varepsilon_o > 0$ such that, for each $\varepsilon \in (0, \varepsilon_o)$, the natural morphisms*

$$Rq_{\partial\varepsilon,*} F^*|_{Y \times S_\varepsilon \times S} \longleftarrow Rq_{\overline{\varepsilon},*} Rj_* F^* \longrightarrow Rq_{\varepsilon,*} Rj_* F^* \longrightarrow i^{-1} Rj_* F^*$$

are isomorphisms.

Proof. We note that, according to Corollary 2.8, $F := Rj_* F^*$ is S -weakly \mathbb{R} -constructible, and is adapted to the stratification $(Y \times Z_\alpha)$. On the other hand, according to §2.1, it is enough to consider the case where F^* is an object of $D_{w\text{-}\mathbb{R}\text{-c}}^b(p_{X^*}^{-1} \mathbb{C}_S)$.

Let us start with the right morphisms. We can argue with any object $F \in D_{w\text{-}\mathbb{R}\text{-c}}^b(p_X^{-1} \mathbb{C}_S)$, not necessarily of the form $Rj_* F^*$. Recall that we have

an adjunction morphism $q_\varepsilon^{-1}Rq_{\varepsilon,*} \rightarrow \text{id}$ and thus $i^{-1}q_\varepsilon^{-1}Rq_{\varepsilon,*} \rightarrow i^{-1}$. Since $q_\varepsilon \circ i = \text{id}_{Y \times S}$, we get the second right morphism. The first one is the restriction morphism.

According to [5, Prop. 8.3.12 and 5.4.17], there exists $\varepsilon_o > 0$ such that, for $\varepsilon' < \varepsilon$ in $(0, \varepsilon_o)$, the restriction morphisms $Rq_{\varepsilon,*}F \rightarrow Rq_{\varepsilon,*}F \rightarrow Rq_{\varepsilon',*}F \rightarrow Rq_{\varepsilon',*}F$ are isomorphisms. In particular, the first right morphism is an isomorphism.

Let us take a q -soft representative of F , that we still denote by F . The inductive system $q_{\varepsilon,*}F$ ($\varepsilon \rightarrow 0$) has limit $i^{-1}F$ and all morphisms of this system are quasi-isomorphisms. Hence the second right morphism is a quasi-isomorphism.

Remark 2.12. A similar argument gives an isomorphism $i^!F \xrightarrow{\sim} Rq_{\varepsilon,!}F$, by using [5, Prop. 5.4.17(c)].

For the left morphism, we take a q -soft representative of F^* that we still denote by F^* . For $\varepsilon_- < \varepsilon < \varepsilon_+ < \varepsilon_o$, we denote by $B_{\varepsilon_-,\varepsilon_+}$ the open set $B_{\varepsilon_+} \setminus \overline{B_{\varepsilon_-}}$ and by $q_{\varepsilon_-,\varepsilon_+}$ the corresponding projection. We have $q_{\partial\varepsilon,*}F^* = \varinjlim_{|\varepsilon_+ - \varepsilon_-| \rightarrow 0} q_{\varepsilon_-,\varepsilon_+,*}F^*$. On the other hand, the morphisms of this inductive system are all quasi-isomorphisms, according to [5, Prop. 5.4.17]. Fixing $\varepsilon' \in (\varepsilon, \varepsilon_o)$ we find a quasi-isomorphism $q_{\varepsilon',*}F^* \rightarrow q_{\partial\varepsilon,*}F^*$. On the other hand, from the first part we have $q_{\varepsilon',*}F^* \xrightarrow{\sim} q_{\varepsilon,*}F^*$, hence the result. q.e.d.

2.4. S -coherent local systems and S - \mathbb{R} -constructible sheaves.

Notation 2.13. We shall denote by $D_{\text{lc coh}}^b(p_X^{-1}\mathcal{O}_S)$ the full triangulated subcategory of $D_{\text{lc}}^b(p_X^{-1}\mathcal{O}_S)$ whose objects satisfy, locally on X , $F \simeq p_X^{-1}G$ with $G \in D_{\text{coh}}^b(\mathcal{O}_S)$. Equivalently, for each $x \in X$, $F|_{\{x\} \times S} \in D_{\text{coh}}^b(\mathcal{O}_S)$ (see the remarks before Lemma 2.3).

Definition 2.14. Given $F \in D_{\text{w-}\mathbb{R}\text{-c}}^b(p_X^{-1}\mathcal{O}_S)$, we say that F is \mathbb{R} -constructible if, for some μ -stratification of X , $X = \bigsqcup_\alpha X_\alpha$, for all $j \in \mathbb{Z}$, $\mathcal{H}^j(F)|_{X_\alpha \times S} \in D_{\text{lc coh}}^b(p_{X_\alpha}^{-1}\mathcal{O}_S)$. This condition characterizes a full triangulated subcategory of $D_{\text{w-}\mathbb{R}\text{-c}}^b(p_X^{-1}\mathcal{O}_S)$ which we denote by $D_{\mathbb{R}\text{-c}}^b(p_X^{-1}\mathcal{O}_S)$.

Similarly to Proposition 2.6 we have:

Proposition 2.15. *Let $F \in D_{\mathbb{R}\text{-c}}^b(p_X^{-1}\mathcal{O}_S)$ and let $s_o \in S$. Then $Li_{s_o}^*(F) \in D_{\mathbb{R}\text{-c}}^b(\mathbb{C}_X)$.*

Remark 2.16. An object of $D_{\text{w-}\mathbb{R}\text{-c}}^b(p_X^{-1}\mathcal{O}_S)$ is in $D_{\mathbb{R}\text{-c}}^b(p_X^{-1}\mathcal{O}_S)$ if and only if, for any $x \in X$, $F|_{\{x\} \times S}$ belongs to $D_{\text{coh}}^b(\mathcal{O}_S)$.

A straightforward adaptation of [5, Prop. 8.4.8] gives:

Proposition 2.17. *Let $f : Y \rightarrow X$ be a morphism of manifolds and let $F \in D_{\mathbb{R}\text{-c}}^b(p_Y^{-1}\mathcal{O}_S)$. Assume that f_S is proper on $\text{Supp}(F)$. Then*

$$Rf_{S,*}F \in D_{\mathbb{R}\text{-c}}^b(p_X^{-1}\mathcal{O}_S).$$

We can also characterize $D_{\mathbb{R}\text{-c}}^b(p_X^{-1}\mathcal{O}_S)$ as in Corollary 2.9.

Corollary 2.18. *An object $F \in D^b(p_X^{-1}\mathcal{O}_S)$ is in $D_{\mathbb{R}\text{-c}}^b(p_X^{-1}\mathcal{O}_S)$ if and only if, for some subanalytic Whitney stratification (X_α) of X , the complexes $i_\alpha^!F$ belong to $D_{\text{lc coh}}^b(p_\alpha^{-1}\mathcal{O}_S)$.*

Proof. Assume F is in $D_{\mathbb{R}\text{-c}}^b(p_X^{-1}\mathcal{O}_S)$. We need to prove the coherence of $i_\alpha^!F$. We argue by induction as in Corollary 2.9, with the same notation. Since the question is local on X_k , by the Whitney property of the stratification (X_α) we can assume that $X_{k-1} = Z \times Y_k$ and there exists a Whitney stratification (Z_α) of Z such that $X_\alpha = Z_\alpha \times Y_k$ for each α such that $X_\alpha \subset X_{k-1}$ (see e.g. [1, §1.4]). Proving that $i_k^!F$ is $p^{-1}\mathcal{O}_S$ -coherent is equivalent to proving that $i_k^{-1}Rj_{k,*}j_k^{-1}F$ is so, since we already know that $i_k^{-1}F$ is so. According to Lemma 2.11, $i_k^{-1}Rj_{k,*}j_k^{-1}F$ is computed as $Rq_{\partial\varepsilon,*}j_k^{-1}F$, and since $q_{\partial\varepsilon}$ is proper, we can apply Proposition 2.17 to get the coherence.

Conversely, Corollary 2.9 already implies that F is an object of $D_{\mathbb{W}\text{-}\mathbb{R}\text{-c}}^b(p_X^{-1}\mathcal{O}_S)$. We argue then as above: since we know by assumption that $i_k^!F$ is coherent, it suffices to prove that $i_k^{-1}Rj_{k,*}j_k^{-1}F$ is so, and the previous argument applies. q.e.d.

2.5. S -weakly \mathbb{C} -constructible sheaves and S - \mathbb{C} -constructible sheaves. Let now assume that X is a complex analytic manifold.

Definition 2.19.

- (1) Let $F \in D_{\mathbb{W}\text{-}\mathbb{R}\text{-c}}^b(p_X^{-1}\mathbb{C}_S)$ (resp. $F \in D_{\mathbb{W}\text{-}\mathbb{R}\text{-c}}^b(p_X^{-1}\mathcal{O}_S)$). We shall say that F is S -weakly \mathbb{C} -constructible if $SS(F)$ is \mathbb{C}^* -conic. The corresponding categories are denoted by $D_{\mathbb{W}\text{-}\mathbb{C}\text{-c}}^b(p_X^{-1}\mathbb{C}_S)$ (resp. $F \in D_{\mathbb{W}\text{-}\mathbb{C}\text{-c}}^b(p_X^{-1}\mathcal{O}_S)$).
- (2) If F belongs to $D_{\mathbb{W}\text{-}\mathbb{C}\text{-c}}^b(p_X^{-1}\mathcal{O}_S)$, we say that F is S - \mathbb{C} -constructible if $F \in D_{\mathbb{R}\text{-c}}^b(p_X^{-1}\mathcal{O}_S)$, and we denote by $D_{\mathbb{C}\text{-c}}^b(p_X^{-1}\mathcal{O}_S)$ the corresponding category, which is full triangulated sub-category of $D^b(p_X^{-1}\mathcal{O}_S)$.

The following properties are obtained in a straightforward way, by using [5, Th. 8.5.5] in a way similar to [5, Prop. 8.5.7].

Properties 2.20.

- (1) An object F of $D^b(p_X^{-1}\mathcal{O}_S)$ belongs to $D_{\mathbb{W}\text{-}\mathbb{C}\text{-c}}^b(p_X^{-1}\mathcal{O}_S)$ if and only if it belongs to $D_{\mathbb{W}\text{-}\mathbb{C}\text{-c}}^b(p_X^{-1}\mathbb{C}_S)$.
- (2) Remark 2.16 applies to $D_{\mathbb{W}\text{-}\mathbb{C}\text{-c}}^b(p_X^{-1}\mathcal{O}_S)$ and $D_{\mathbb{C}\text{-c}}^b(p_X^{-1}\mathcal{O}_S)$.
- (3) Proposition 2.7 applies to $D_{\mathbb{W}\text{-}\mathbb{C}\text{-c}}^b$.
- (4) Propositions 2.15, 2.17, and Corollary 2.18 apply to $D_{\mathbb{C}\text{-c}}^b(p_X^{-1}\mathcal{O}_S)$.
- (5) Corollary 2.10 applies to $D_{\mathbb{W}\text{-}\mathbb{C}\text{-c}}^b$, $D_{\mathbb{R}\text{-c}}^b$ and $D_{\mathbb{C}\text{-c}}^b$.

2.6. Duality. According to the syzygy theorem for the regular local ring $\mathcal{O}_{S,s}$ (for any $s \in S$) and e.g. [6, Prop. 13.2.2(ii)] (for the opposite category), any object of $D_{\text{coh}}^b(\mathcal{O}_S)$ is locally quasi-isomorphic to a bounded complex of locally free \mathcal{O}_S -modules of finite rank L^\bullet . As a consequence, the local duality functor

$$D : D_{\text{coh}}^b(\mathcal{O}_S) \rightarrow D_{\text{coh}}^b(\mathcal{O}_S), \quad D(\mathcal{F}) := R\mathcal{H}om_{\mathcal{O}_S}(\mathcal{F}, \mathcal{O}_S)$$

is seen to be an involution, i.e., the natural morphism $\text{id} \rightarrow \mathbf{D} \circ \mathbf{D}$ is an isomorphism. However, the standard t-structure

$$(\mathbf{D}_{\text{coh}}^{\text{b}, \leq 0}(\mathcal{O}_S), \mathbf{D}_{\text{coh}}^{\text{b}, \geq 0}(\mathcal{O}_S))$$

defined by $\mathcal{H}^j G = 0$ for $j > 0$ (resp. for $j < 0$) is not interchanged by duality when $\dim S \geq 1$ (see e.g., [4, Prop. 4.3] in the algebraic setting). Nevertheless, we have:

Lemma 2.21. *Let G be an object of $\mathbf{D}_{\text{coh}}^{\text{b}}(\mathcal{O}_S)$. Assume that $\mathbf{D}G$ belongs to $\mathbf{D}_{\text{coh}}^{\text{b}, \leq 0}(\mathcal{O}_S)$. Then G belongs to $\mathbf{D}_{\text{coh}}^{\text{b}, \geq 0}(\mathcal{O}_S)$.*

Proof. Setting $G' = \mathbf{D}G$, the biduality isomorphism makes it equivalent to proving that $\mathbf{D}G'$ belongs to $\mathbf{D}_{\text{coh}}^{\text{b}, \geq 0}(\mathcal{O}_S)$. The question is local on S and we may therefore replace G' with a bounded complex L^\bullet as above. Moreover, L^\bullet is quasi-isomorphic to such a bounded complex, still denoted by L^\bullet , such that $L^k = 0$ for $k > 0$. Indeed, note first that the kernel K of a surjective morphism of locally free \mathcal{O}_S -modules of finite rank is also locally free of finite rank (being \mathcal{O}_S -coherent and having all its germs K_s free over $\mathcal{O}_{S,s}$, because they are projective and $\mathcal{O}_{S,s}$ is a regular local ring). By assumption, we have $\mathcal{H}^j(L^\bullet) = 0$ for $j > 0$. Let $k > 0$ be such that $L^k \neq 0$ and $L^\ell = 0$ for $\ell > k$, and let $L'^{k-1} = \ker[L^{k-1} \rightarrow L^k]$. Then L^\bullet is quasi-isomorphic to L'^\bullet defined by $L'^j = L^j$ for $j < k-1$ and $L'^j = 0$ for $j \geq k$. We conclude by induction on k .

Now it is clear that $\mathbf{D}G' \simeq \mathbf{D}L^\bullet$ is a bounded complex having terms in nonnegative degrees at most, and thus is an object of $\mathbf{D}_{\text{coh}}^{\text{b}, \geq 0}(\mathcal{O}_S)$. q.e.d.

Remark 2.22. Let G be an object of $\mathbf{D}_{\text{coh}}^{\text{b}}(\mathcal{O}_S)$. Assume that G and $\mathbf{D}G$ belong to $\mathbf{D}_{\text{coh}}^{\text{b}, \leq 0}(\mathcal{O}_S)$. Then G and $\mathbf{D}G$ are \mathcal{O}_S -coherent sheaves, hence G and $\mathbf{D}G$ are \mathcal{O}_S -locally free.

We now set $\omega_{X,S} = p_X^{-1} \mathcal{O}_S[2 \dim X] = p_X^! \mathcal{O}_S$.

Proposition 2.23. *The functor $\mathbf{D} : \mathbf{D}^{\text{b}}(p_X^{-1} \mathcal{O}_S) \rightarrow \mathbf{D}^+(p_X^{-1} \mathcal{O}_S)$ defined by $\mathbf{D}F = R\mathcal{H}om_{p_X^{-1} \mathcal{O}_S}(F, \omega_{X,S})$ induces an involution $\mathbf{D}_{\mathbb{R}\text{-c}}^{\text{b}}(p_X^{-1} \mathcal{O}_S) \rightarrow \mathbf{D}_{\mathbb{R}\text{-c}}^{\text{b}}(p_X^{-1} \mathcal{O}_S)$ and $\mathbf{D}_{\mathbb{C}\text{-c}}^{\text{b}}(p_X^{-1} \mathcal{O}_S) \rightarrow \mathbf{D}_{\mathbb{C}\text{-c}}^{\text{b}}(p_X^{-1} \mathcal{O}_S)$.*

We will also set $\mathbf{D}'F = R\mathcal{H}om_{p_X^{-1} \mathcal{O}_S}(F, p_X^{-1} \mathcal{O}_S)$.

Proof. Let us first show that, for F in $\mathbf{D}_{\text{w-}\mathbb{R}\text{-c}}^{\text{b}}(p_X^{-1} \mathcal{O}_S)$, the dual $\mathbf{D}F$ also belongs to $\mathbf{D}_{\text{w-}\mathbb{R}\text{-c}}^{\text{b}}(p_X^{-1} \mathcal{O}_S)$. Let (X_α) be a μ -stratification adapted to F . According to Corollary 2.9, it is enough to show that $i_\alpha^! \mathbf{D}F$ has locally constant cohomology for each α . One can use [5, Prop. 3.1.13] in our setting and get

$$i_\alpha^! \mathbf{D}F = R\mathcal{H}om_{p_\alpha^{-1} \mathcal{O}_S}(i_\alpha^{-1} F, \omega_{X_\alpha, S}).$$

Locally on X_α , $i_\alpha^{-1} F = p_\alpha^{-1} G$ for some G in $\mathbf{D}^{\text{b}}(\mathbb{C}_S)$ or $\mathbf{D}^{\text{b}}(\mathcal{O}_S)$. Then, locally on X_α ,

$$\begin{aligned} i_\alpha^! \mathbf{D}F &\simeq R\mathcal{H}om_{p_\alpha^{-1} \mathcal{O}_S}(p_\alpha^{-1} G, p_\alpha^! \mathcal{O}_S) = p_\alpha^! R\mathcal{H}om_{\mathcal{O}_S}(G, \mathcal{O}_S) \\ &= p_\alpha^{-1}(\mathbf{D}G)[2 \dim X_\alpha]. \end{aligned}$$

The proof for F in $D_{w\text{-}\mathbb{C}\text{-c}}^b(p_X^{-1}\mathcal{O}_S)$ is similar. Moreover, by using Corollary 2.18 instead of Corollary 2.9 one shows that \mathbf{D} sends $D_{\mathbb{R}\text{-c}}^b(p_X^{-1}\mathcal{O}_S)$ to itself and, according to Properties 2.20(4), $D_{\mathbb{C}\text{-c}}^b(p_X^{-1}\mathcal{O}_S)$ to itself.

Let us prove the involution property. We have a natural morphism of functors $\text{id} \rightarrow \mathbf{D}\mathbf{D}$. It is enough to prove the isomorphism property after applying $Li_{s_o}^*$ for each $s_o \in S$, according to Proposition 2.2. On the other hand, Proposition 2.1 implies that $Li_{s_o}^*$ commutes with \mathbf{D} , so we are reduced to applying the involution property on $D_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X)$, according to the \mathbb{C} -c-analogue of Proposition 2.15, which is known to be true (see e.g. [5]). q.e.d.

Remark 2.24. By using the biduality isomorphism and the isomorphism $i_x^! \mathbf{D}F \simeq \mathbf{D}i_x^{-1}F$ for F in $D_{\mathbb{R}\text{-c}}^b(p_X^{-1}\mathcal{O}_S)$ or $D_{\mathbb{C}\text{-c}}^b(p_X^{-1}\mathcal{O}_S)$, where $i_x : \{x\} \times S \hookrightarrow X \times S$ denotes the inclusion, we find a functorial isomorphism $i_x^{-1} \mathbf{D}F \simeq \mathbf{D}i_x^! F$.

2.7. Perversity. We will now restrict to the case of S - \mathbb{C} -constructible complexes, which is the only case which will be of interest for us, although one could consider the case of S - \mathbb{R} -constructible complexes as in [5, §10.2].

We define the category ${}^pD_{\mathbb{C}\text{-c}}^{\leq 0}(p_X^{-1}\mathcal{O}_S)$ as the full subcategory of $D_{\mathbb{C}\text{-c}}^b(p_X^{-1}\mathcal{O}_S)$ whose objects are the S - \mathbb{C} -constructible bounded complexes F such that, for some adapted μ -stratification (X_α) (i_x is as above),

$$(\text{Supp}) \quad \forall \alpha, \forall x \in X_\alpha, \forall j > -\dim X_\alpha, \quad \mathcal{H}^j i_x^{-1} F = 0.$$

Similarly, ${}^pD_{\mathbb{C}\text{-c}}^{\geq 0}(p_X^{-1}\mathcal{O}_S)$ consists of objects F such that

$$(\text{Cosupp}) \quad \forall \alpha, \forall x \in X_\alpha, \forall j < \dim X_\alpha, \quad \mathcal{H}^j i_x^! F = 0.$$

In the preceding situation in view of Corollary 2.18 we have, similarly to [5, Prop.10.2.4]:

Lemma 2.25.

- (1) $F \in {}^pD_{\mathbb{C}\text{-c}}^{\leq 0}(p_X^{-1}\mathcal{O}_S)$ if and only if for any α and $j > -\dim(X_\alpha)$,

$$\mathcal{H}^j(i_\alpha^{-1}F) = 0.$$
- (2) $F \in {}^pD_{\mathbb{C}\text{-c}}^{\geq 0}(p_X^{-1}\mathcal{O}_S)$ if and only if for any α and $j < -\dim(X_\alpha)$,

$$\mathcal{H}^j(i_\alpha^!F) = 0.$$

Namely, if $F \in {}^pD_{\mathbb{C}\text{-c}}^{\leq 0}(p_X^{-1}\mathcal{O}_S)$ and Z is a closed analytic subset of X such that $\dim Z = k$, then $i_{Z \times S}^{-1}F$ is concentrated in degrees $\leq -k$, and if $F' \in {}^pD_{\mathbb{C}\text{-c}}^{\geq 0}(p_X^{-1}\mathcal{O}_S)$, then $i_{Z \times S}^!F'$ is concentrated in degrees $\geq -k$. We have the following variant of [5, Prop.10.2.7]:

Proposition 2.26. *Let F be an object of ${}^pD_{w\text{-}\mathbb{R}\text{-c}}^{\leq 0}(p_X^{-1}\mathcal{O}_S)$ and F' an object of ${}^pD_{w\text{-}\mathbb{R}\text{-c}}^{\geq 0}(p_X^{-1}\mathcal{O}_S)$. Then*

$$\mathcal{H}^j R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F, F') = 0, \quad \text{for } j < 0.$$

Proof. Let (X_α) be a μ -stratification of X adapted to F and F' . By assumption, for each α , $i_\alpha^{-1} \mathcal{H}^j F = \mathcal{H}^j i_\alpha^{-1} F = 0$ for $j > -\dim X_\alpha$. Similarly, $\mathcal{H}^j i_\alpha^! F' = 0$ for $j < -\dim X_\alpha$.

Let X_α be a stratum of maximal dimension such that

$$i_\alpha^{-1} \mathcal{H}^j R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F, F') \neq 0 \quad \text{for some } j < 0.$$

Let V be an open neighbourhood of X_α in X such that $V \setminus X_\alpha$ intersects only strata of dimension $> \dim X_\alpha$, and let $j_\alpha : (V \setminus X_\alpha) \times S \hookrightarrow V \times S$ be the inclusion. Then the complex $i_\alpha^{-1} Rj_{\alpha,*} j_\alpha^{-1} R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F, F')$ has nonzero cohomology in nonnegative degrees only: indeed, by the definition of X_α , this property holds for $j_\alpha^{-1} R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F, F')$, hence it holds for $Rj_{\alpha,*} j_\alpha^{-1} R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F, F')$, and then clearly for the complex $i_\alpha^{-1} Rj_{\alpha,*} j_\alpha^{-1} R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F, F')$. From the distinguished triangle

$$\begin{aligned} i_\alpha^! R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F, F') &\rightarrow i_\alpha^{-1} R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F, F') \\ &\rightarrow i_\alpha^{-1} Rj_{\alpha,*} j_\alpha^{-1} R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F, F') \xrightarrow{+1} \end{aligned}$$

we conclude that $\mathcal{H}^j i_\alpha^! R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F, F') \rightarrow \mathcal{H}^j i_\alpha^{-1} R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F, F') = i_\alpha^{-1} \mathcal{H}^j R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F, F')$ is an isomorphism for all $j < 0$. Therefore, we obtain, for this stratum X_α and for any $j < 0$,

$$\begin{aligned} i_\alpha^{-1} \mathcal{H}^j R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F, F') &\simeq \mathcal{H}^j i_\alpha^! R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F, F') \\ &\simeq \mathcal{H}^j R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(i_\alpha^{-1} F, i_\alpha^! F'). \end{aligned}$$

Since $i_\alpha^{-1} F$ has nonzero cohomology in degrees $\leq -\dim X_\alpha$ at most and $i_\alpha^! F'$ in degrees $\geq -\dim X_\alpha$ at most, $\mathcal{H}^j R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(i_\alpha^{-1} F, i_\alpha^! F') = 0$ for $j < 0$, a contradiction with the definition of X_α . q.e.d.

Theorem 2.27. $\mathrm{PD}_{\mathbb{C}\text{-c}}^{\leq 0}(p_X^{-1}\mathcal{O}_S)$ and $\mathrm{PD}_{\mathbb{C}\text{-c}}^{\geq 0}(p_X^{-1}\mathcal{O}_S)$ form a t -structure of $\mathrm{D}_{\mathbb{C}\text{-c}}^b(p_X^{-1}\mathcal{O}_S)$, whose heart is denoted by $\mathrm{Perv}(p_X^{-1}\mathcal{O}_S)$.

Sketch of proof. We have to prove:

- (1) $\mathrm{PD}_{\mathbb{C}\text{-c}}^{\leq 0} \subset \mathrm{PD}_{\mathbb{C}\text{-c}}^{\leq 1}$ and $\mathrm{PD}_{\mathbb{C}\text{-c}}^{\geq 0} \supset \mathrm{PD}_{\mathbb{C}\text{-c}}^{\geq 1}$.
- (2) For $F \in \mathrm{PD}_{\mathbb{C}\text{-c}}^{\leq 0}(p_X^{-1}\mathcal{O}_S)$ and $F' \in \mathrm{PD}_{\mathbb{C}\text{-c}}^{\geq 1}(p_X^{-1}\mathcal{O}_S)$,

$$\mathrm{Hom}_{\mathrm{D}^b(p_X^{-1}\mathcal{O}_S)}(F, F') = 0.$$

- (3) For any $F \in \mathrm{D}_{\mathbb{C}\text{-c}}^b(p_X^{-1}\mathcal{O}_S)$ there exist $F' \in \mathrm{PD}_{\mathbb{C}\text{-c}}^{\leq 0}(p_X^{-1}\mathcal{O}_S)$ and $F'' \in \mathrm{PD}_{\mathbb{C}\text{-c}}^{\geq 1}(p_X^{-1}\mathcal{O}_S)$, giving rise to a distinguished triangle $F' \rightarrow F \rightarrow F'' \xrightarrow{+1}$.

Then, following the line of the proof of [5, Theorem 10.2.8], we observe that (1) is obvious and (2) follows from Proposition 2.26. Now, (3) is deduced by mimicking stepwise the proof of (c) in [5, Theorem 10.2.8]. q.e.d.

According to the preliminary remarks before Lemma 2.21, one cannot expect that the previous t -structure is interchanged by duality when $\dim S \geq 1$. However we have:

Proposition 2.28. *Let F be an object of $\mathrm{PD}_{\mathbb{C}\text{-c}}^{\leq 0}(p_X^{-1}\mathcal{O}_S)$ such that $\mathbf{D}F$ also belongs to $\mathrm{PD}_{\mathbb{C}\text{-c}}^{\leq 0}(p_X^{-1}\mathcal{O}_S)$. Then F and $\mathbf{D}F$ are objects of $\mathrm{Perv}(p_X^{-1}\mathcal{O}_S)$.*

Proof. Let us fix $x \in X_\alpha$. We have $i_x^! F \simeq \mathbf{D}(i_x^{-1} \mathbf{D}F)$, as already observed in Remark 2.24. By assumption $G := i_x^{-1} \mathbf{D}F$ belongs to $\mathbf{D}_{\text{coh}}^{\text{b}, \leq -\dim X_\alpha}(\mathcal{O}_S)$, and Lemma 2.21 suitably shifted and applied to $\mathbf{D}G$ implies that $\mathbf{D}G$ belongs to $\mathbf{D}_{\text{coh}}^{\text{b}, \geq \dim X_\alpha}(\mathcal{O}_S)$, which is the cosupport condition (Cosupp) for F . q.e.d.

Assume $F \in \text{Perv}(p_X^{-1} \mathcal{O}_S)$. The description of the dual standard t-structure on $\mathbf{D}_{\text{coh}}^{\text{b}}(\mathcal{O}_S)$ given in [4, §4] supplies the following refinement to (Supp) and (Cosupp) when $\mathbf{D}F$ is also perverse.

Corollary 2.29. *Let $F \in \text{Perv}(p_X^{-1} \mathcal{O}_S)$ and assume that $\mathbf{D}F \in \text{Perv}(p_X^{-1} \mathcal{O}_S)$. Let (X_α) be a stratification adapted to F . Then for each α , each $x \in X_\alpha$ and each closed analytic subset $Z \subset S$, we have*

$$(\text{Cosupp}+) \quad \mathcal{H}^k(i_{Z \times \{x\}}^! F) = 0, \quad \forall k < \text{codim}_S Z + \dim X_\alpha.$$

(The perversity of F only gives the previous property when $Z = S$.)

3. THE DE RHAM COMPLEX OF A HOLONOMIC $\mathcal{D}_{X \times S/S}$ -MODULE

In what follows X and S denote complex manifolds and we set $n = \dim X$, $\ell = \dim S$. We shall keep the notation of the preceding section. Let $\pi : T^*(X \times S) \rightarrow T^*X \times S$ denote the projection and let $\mathcal{D}_{X \times S/S}$ denote the subsheaf of $\mathcal{D}_{X \times S}$ of relative differential operators with respect to p_X (see [18, §2.1 & 2.2]).

Recall that $p_X^{-1} \mathcal{O}_S$ is contained in the center of $\mathcal{D}_{X \times S/S}$. With the same proof as for Proposition 2.1 we obtain:

Proposition 3.1. *Let $s_o \in S$ be given. Let \mathcal{M} and \mathcal{N} be objects of $\mathbf{D}^{\text{b}}(\mathcal{D}_{X \times S/S})$. Then, there is a well-defined natural morphism*

$$Li_{s_o}^*(R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{M}, \mathcal{N})) \rightarrow R\mathcal{H}om_{i_{s_o}^*(\mathcal{D}_{X \times S/S})}(Li_{s_o}^*(\mathcal{M}), Li_{s_o}^*(\mathcal{N}))$$

which is an isomorphism in $\mathbf{D}^{\text{b}}(\mathbb{C}_X)$.

3.1. Duality for coherent $\mathcal{D}_{X \times S/S}$ -modules. We refer for instance to [3, Appendix] for the coherence properties of the ring $\mathcal{D}_{X \times S/S}$. The classical methods used in the absolute case, i.e, for coherent \mathcal{D}_X -objects (see for instance [8, Prop. 2.1.16], [9, Prop. 2.7-3]) apply here:

Proposition 3.2. *Let \mathcal{M} be a coherent $\mathcal{D}_{X \times S/S}$ -module. Then \mathcal{M} locally admits a resolution of length at most $2n + \ell$ by free $\mathcal{D}_{X \times S/S}$ -modules of finite rank.*

Proposition 3.2 and [6, Prop. 13.2.2(ii)] (for the opposite category) imply:

Corollary 3.3. *Let $\mathcal{M} \in \mathbf{D}_{\text{coh}}^{\text{b}}(\mathcal{D}_{X \times S/S})$. Let us assume that \mathcal{M} is concentrated in degrees $[a, b]$. Then, in a neighborhood of each $(x, z) \in X \times S$, there exist a complex \mathcal{L}^\bullet of free $\mathcal{D}_{X \times S/S}$ -modules of finite rank concentrated in degrees $[a - 2n - \ell, b]$ and a quasi-isomorphism $\mathcal{L}^\bullet \rightarrow \mathcal{M}$.*

We set $\Omega_{X \times S/S} = \Omega_{X \times S/S}^n$, where $\Omega_{X \times S/S}^n$ denotes the sheaf of relative differential forms of degree $n = \dim X$.

Definition 3.4. The duality functor $\mathbf{D}(\cdot) : \mathbf{D}^b(\mathcal{D}_{X \times S/S}) \rightarrow \mathbf{D}^b(\mathcal{D}_{X \times S/S})$ is defined as:

$$\mathcal{M} \mapsto \mathbf{D}\mathcal{M} = R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{M}, \mathcal{D}_{X \times S/S} \otimes_{\mathcal{O}_{X \times S}} \Omega_{X \times S/S}^{\otimes -1})[n].$$

We also set $\mathbf{D}'\mathcal{M} := R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{M}, \mathcal{D}_{X \times S/S}) \in \mathbf{D}^b(\mathcal{D}_{X \times S/S}^{\text{opp}})$.

By Proposition 3.2, $\mathcal{D}_{X \times S/S}$ has finite cohomological dimension, so [3, (A.11)] gives a natural morphism in $\mathbf{D}^b(\mathcal{D}_{X \times S/S})$:

$$(1) \quad \mathcal{M} \rightarrow \mathbf{D}'\mathbf{D}'\mathcal{M} \simeq \mathbf{D}\mathbf{D}\mathcal{M}.$$

Moreover, in view of Corollary 3.3, if $\mathcal{M} \in \mathbf{D}_{\text{coh}}^b(\mathcal{D}_{X \times S/S})$, then $\mathbf{D}'\mathcal{M} \in \mathbf{D}_{\text{coh}}^b(\mathcal{D}_{X \times S/S}^{\text{opp}})$. Indeed, we may choose a local free finite resolution \mathcal{L}^\bullet of \mathcal{M} , so that $\mathbf{D}'\mathcal{M}$ is quasi isomorphic to the transposed complex $(\mathcal{L}^\bullet)^t$ whose entries are free.

By the same argument we deduce that (1) is an isomorphism whenever $\mathcal{M} \in \mathbf{D}_{\text{coh}}^b(\mathcal{D}_{X \times S/S})$.

Again by Proposition 3.2, $\mathcal{D}_{X \times S/S}$ has finite flat dimension so we are in conditions to apply [3, (A.10)]: given $\mathcal{M}, \mathcal{N} \in \mathbf{D}^b(\mathcal{D}_{X \times S/S})$ there is a natural morphism:

$$(2) \quad \mathbf{D}'\mathcal{M} \overset{L}{\otimes}_{\mathcal{D}_{X \times S/S}} \mathcal{N} \rightarrow R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{M}, \mathcal{N})$$

which is an isomorphism provided that \mathcal{M} or \mathcal{N} belong to $\mathbf{D}_{\text{coh}}^b(\mathcal{D}_{X \times S/S})$. When $\mathcal{M}, \mathcal{N} \in \mathbf{D}_{\text{coh}}^b(\mathcal{D}_{X \times S/S})$, composing (2) with the biduality isomorphism (1) gives a natural isomorphism

$$(3) \quad R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{M}, \mathcal{N}) \simeq R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathbf{D}\mathcal{N}, \mathbf{D}\mathcal{M}).$$

3.2. Characteristic variety. Recall (see [17, §III.1.3]) that the characteristic variety $\text{Char } \mathcal{M}$ of a coherent $\mathcal{D}_{X \times S/S}$ -module \mathcal{M} is the support in $T^*X \times S$ of its graded module with respect to any (local) good filtration. One has (see [17, Prop. III.1.3.2])

$$(4) \quad \begin{aligned} \text{Char}(\mathcal{D}_{X \times S} \otimes_{\mathcal{D}_{X \times S/S}} \mathcal{M}) &= \pi^{-1} \text{Char } \mathcal{M}, \\ \text{Char } \mathcal{M} &= \pi(\text{Char}(\mathcal{D}_{X \times S} \otimes_{\mathcal{D}_{X \times S/S}} \mathcal{M})). \end{aligned}$$

One may as well define the characteristic variety of an object $\mathcal{M} \in \mathbf{D}_{\text{coh}}^b(\mathcal{D}_{X \times S/S})$ as the union of the characteristic varieties of its cohomology modules. By the flatness of $\mathcal{D}_{X \times S}$ over $\mathcal{D}_{X \times S/S}$, (4) holds for any object of $\mathbf{D}_{\text{coh}}^b(\mathcal{D}_{X \times S/S})$.

Proposition 3.5 ([18, Prop. 2.5]). *For $\mathcal{M} \in \mathbf{D}_{\text{coh}}^b(\mathcal{D}_{X \times S/S})$ we have*

$$\text{Char}(\mathcal{M}) = \text{Char}(\mathbf{D}\mathcal{M}).$$

3.3. The de Rham and solution complexes. For an object \mathcal{M} of $\mathbf{D}^b(\mathcal{D}_{X \times S/S})$ we define the functors

$$\begin{aligned} \text{DR } \mathcal{M} &:= R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{O}_{X \times S}, \mathcal{M}), \\ \text{Sol } \mathcal{M} &:= R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{M}, \mathcal{O}_{X \times S}) \end{aligned}$$

which take values in $D^b(p_X^{-1}\mathcal{O}_S)$. If \mathcal{M} is a $\mathcal{D}_{X \times S/S}$ -module, that is, a $\mathcal{O}_{X \times S}$ -module equipped with an integrable relative connection $\nabla : \mathcal{M} \rightarrow \Omega_{X \times S/S}^1 \otimes \mathcal{M}$, the object $\mathrm{DR} \mathcal{M}$ is represented by the complex $(\Omega_{X \times S/S}^\bullet \otimes_{\mathcal{O}_{X \times S}} \mathcal{M}, \nabla)$.

Noting that $R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{O}_{X \times S}, \mathcal{D}_{X \times S/S}) \simeq \Omega_{X \times S/S}[-\dim X]$ we get

$$D\mathcal{O}_{X \times S} \simeq \mathcal{O}_{X \times S}.$$

For $\mathcal{N} = \mathcal{O}_{X \times S}$, (3) implies a natural isomorphism, for $\mathcal{M} \in D_{\mathrm{coh}}^b(\mathcal{D}_{X \times S/S})$:

$$(5) \quad \mathrm{Sol} \mathcal{M} \simeq \mathrm{DR} D\mathcal{M}.$$

3.4. Holonomic $\mathcal{D}_{X \times S/S}$ -modules. Let \mathcal{M} be a coherent $\mathcal{D}_{X \times S/S}$ -module. We say that it is *holonomic* if its characteristic variety $\mathrm{Char} \mathcal{M} \subset T^*X \times S$ is contained in $\Lambda \times S$ for some closed conic Lagrangian complex analytic subset of T^*X . We will say that a complex μ -stratification (X_α) is adapted to \mathcal{M} if $\Lambda \subset \bigcup_\alpha T_{X_\alpha}^*X$. Similar definitions hold for objects of $D_{\mathrm{hol}}^b(\mathcal{D}_{X \times S/S})$.

An object $\mathcal{M} \in D_{\mathrm{coh}}^b(\mathcal{D}_{X \times S/S})$ is said to be holonomic if its cohomology modules are holonomic. We denote the full triangulated category of holonomic complexes by $D_{\mathrm{hol}}^b(\mathcal{D}_{X \times S/S})$.

Corollary 3.6 (of Prop. 3.5). *If \mathcal{M} is an object of $D_{\mathrm{hol}}^b(\mathcal{D}_{X \times S/S})$, then so is $D\mathcal{M}$.*

Theorem 3.7. *Let \mathcal{M} be an object of $D_{\mathrm{hol}}^b(\mathcal{D}_{X \times S/S})$. Then $\mathrm{DR}(\mathcal{M})$ and $\mathrm{Sol} \mathcal{M}$ belong to $D_{\mathrm{C-c}}^b(p_X^{-1}\mathcal{O}_S)$.*

Proof. Firstly, it follows [5, Prop. 11.3.3], that $\mathrm{Sol}(\mathcal{M})$ and $\mathrm{DR}(\mathcal{M})$ have their micro-support contained in $\Lambda \times T^*S$ (see [18, p. 11 & Th. 2.13]) and, according to Proposition 2.5, these complexes are objects of $D_{\mathrm{w-C-c}}^b(p_X^{-1}\mathcal{O}_S)$.

Let $x \in X$. In order to prove that $i_x^{-1} \mathrm{DR} \mathcal{M}$ has \mathcal{O}_S -coherent cohomology, we can assume that x is a stratum of a stratification adapted to $\mathrm{DR} \mathcal{M}$ and we use Lemma 2.11 to get $i_x^{-1} \mathrm{DR} \mathcal{M} \simeq Rp_{\bar{\varepsilon},*}(\mathbb{C}_{\bar{B}_\varepsilon \times S} \otimes_{\mathbb{C}} \mathrm{DR} \mathcal{M})$ for ε small enough, where \bar{B}_ε is a closed ball of radius ε centered at x . One then remarks that $(\mathbb{C}_{\bar{B}_\varepsilon \times S}, \mathcal{M})$ forms a relative elliptic pair in the sense of [18], and Proposition 4.1 of loc. cit. gives the desired coherence.

The statement for $\mathrm{Sol} \mathcal{M}$ is proved similarly. q.e.d.

Lemma 3.8 (see [14, Prop. 1.2.5]). *For \mathcal{M} in $D_{\mathrm{hol}}^b(\mathcal{D}_{X \times S/S})$ with adapted stratification (X_α) and for any $s_o \in S$, $Li_{s_o}^* \mathcal{M}$ is \mathcal{D}_X -holonomic and (X_α) is adapted to it.*

Corollary 3.9. *For $\mathcal{M} \in D_{\mathrm{hol}}^b(\mathcal{D}_{X \times S/S})$, there is a natural isomorphism $D' \mathrm{Sol} \mathcal{M} \simeq \mathrm{DR} \mathcal{M}$.*

Proof. We consider the canonical pairing

$$\mathrm{DR} \mathcal{M} \overset{L}{\otimes}_{p_X^{-1}\mathcal{O}_S} \mathrm{Sol} \mathcal{M} \rightarrow p_X^{-1}\mathcal{O}_S$$

which gives a natural morphism

$$\mathrm{DR} \mathcal{M} \rightarrow D' \mathrm{Sol} \mathcal{M}$$

in $D_{\mathbb{C}\text{-c}}^b(p_X^{-1}\mathcal{O}_S)$. We have for each $s_o \in S$, by Proposition 3.1

$$\begin{aligned} Li_{s_o}^*(\mathrm{DR}\mathcal{M}) &\simeq \mathrm{DR} Li_{s_o}^*(\mathcal{M}), \\ Li_{s_o}^*(\mathrm{Sol}\mathcal{M}) &\simeq \mathrm{Sol} Li_{s_o}^*(\mathcal{M}). \end{aligned}$$

Since $Li_{s_o}^*(\mathcal{M}) \in D_{\mathrm{hol}}^b(\mathcal{D}_X)$ by Lemma 3.8, we have

$$\mathrm{DR} Li_{s_o}^*(\mathcal{M}) \simeq \mathbf{D}' \mathrm{Sol} Li_{s_o}^*(\mathcal{M}),$$

so by Proposition 3.1 and Proposition 2.1

$$\mathbf{D}' \mathrm{Sol} Li_{s_o}^*(\mathcal{M}) \simeq \mathbf{D}' Li_{s_o}^*(\mathrm{Sol}\mathcal{M}) \simeq Li_{s_o}^*(\mathbf{D}' \mathrm{Sol}\mathcal{M}).$$

The assertion then follows by Proposition 2.2. q.e.d.

In the following proposition, the main argument is that of strictness, which is essential. We will set ${}^p\mathrm{DR}\mathcal{M} := \mathrm{DR}\mathcal{M}[\dim X]$ and ${}^p\mathrm{Sol}\mathcal{M} = \mathrm{Sol}\mathcal{M}[\dim X]$.

Proposition 3.10. *Let \mathcal{M} be a holonomic $\mathcal{D}_{X \times S/S}$ -module which is strict, i.e., which is $p^{-1}\mathcal{O}_S$ -flat. Then ${}^p\mathrm{DR}\mathcal{M}$ satisfies the support condition (Supp) with respect to a μ -stratification adapted to \mathcal{M} .*

Proof. We prove the result by induction on $\dim S$. Since it is local on S , we consider a local coordinate s on S and we set $S' = \{s = 0\}$. The strictness property implies that we have an exact sequence

$$0 \rightarrow \mathcal{M} \xrightarrow{s} \mathcal{M} \rightarrow i_{S'}^* \mathcal{M} \rightarrow 0,$$

and $i_{S'}^* \mathcal{M}$ is $\mathcal{D}_{X \times S'/S'}$ -holonomic and $p^{-1}\mathcal{O}_{S'}$ -flat. We deduce an exact sequence of complexes $0 \rightarrow {}^p\mathrm{DR}\mathcal{M} \xrightarrow{s} {}^p\mathrm{DR}\mathcal{M} \rightarrow {}^p\mathrm{DR} i_{S'}^* \mathcal{M} \rightarrow 0$.

Let X_α be a stratum of a μ -stratification of X adapted to \mathcal{M} (hence to $i_{S'}^* \mathcal{M}$, after Lemma 3.8). For $x \in X_\alpha$, let k be the maximum of the indices j such that $\mathcal{H}^j i_x^{-1} {}^p\mathrm{DR}\mathcal{M} \neq 0$. For any S' as above, we have a long exact sequence

$$\dots \rightarrow \mathcal{H}^k i_x^{-1} {}^p\mathrm{DR}\mathcal{M} \xrightarrow{s} \mathcal{H}^k i_x^{-1} {}^p\mathrm{DR}\mathcal{M} \rightarrow \mathcal{H}^k i_x^{-1} {}^p\mathrm{DR} i_{S'}^* \mathcal{M} \rightarrow 0.$$

If $k > -\dim X_\alpha$, we have $\mathcal{H}^k i_x^{-1} {}^p\mathrm{DR} i_{S'}^* \mathcal{M} = 0$, according to the support condition for $i_{S'}^* \mathcal{M}$ (inductive assumption), since (X_α) is adapted to it. Therefore, $s : \mathcal{H}^k i_x^{-1} {}^p\mathrm{DR}\mathcal{M} \rightarrow \mathcal{H}^k i_x^{-1} {}^p\mathrm{DR}\mathcal{M}$ is onto. On the other hand, by Theorem 3.7, $\mathcal{H}^k i_x^{-1} {}^p\mathrm{DR}\mathcal{M}$ is \mathcal{O}_S -coherent. Then Nakayama's lemma implies that $\mathcal{H}^k i_x^{-1} {}^p\mathrm{DR}\mathcal{M} = 0$ in some neighbourhood of S' . Since S' was arbitrary, this holds all over S , hence the assertion. q.e.d.

Proof of Theorem 1.2. It is a direct consequence of the following.

Theorem 3.11. *Let \mathcal{M} be an object of $D_{\mathrm{hol}}^b(\mathcal{D}_{X \times S/S})$ and let $\mathbf{D}\mathcal{M}$ be the dual object. Then there is an isomorphism ${}^p\mathrm{DR}\mathbf{D}\mathcal{M} \simeq \mathbf{D} {}^p\mathrm{DR}\mathcal{M}$.*

Indeed, with the assumptions of Theorem 1.2, $\mathbf{D}\mathcal{M}$ is holonomic since \mathcal{M} is so (see Corollary 3.6), and both \mathcal{M} and $\mathbf{D}\mathcal{M}$ are strict. Then both ${}^p\mathrm{DR}\mathcal{M}$ and ${}^p\mathrm{DR}\mathbf{D}\mathcal{M}$ satisfy the support condition, according to Proposition 3.10. Hence, according to Theorem 3.11 and Proposition 2.28, ${}^p\mathrm{DR}\mathcal{M}$ satisfies the cosupport condition.

Similarly, ${}^{\text{p}}\text{Sol } \mathcal{M} \simeq \mathbf{D}^{\text{pDR}} \mathcal{M}$ and $\mathbf{D}({}^{\text{p}}\text{Sol } \mathcal{M}) \simeq {}^{\text{p}}\text{DR } \mathcal{M}$ both satisfy the support condition, hence $\text{Sol } \mathcal{M}[\dim X]$ is a perverse object. q.e.d.

Proof of Theorem 3.11. Combining (3) with [5, Ex. II.24 (iv)] (with $f = \text{id}$, $\mathcal{A} = \mathcal{D}_{X \times S/S}$ and $\mathcal{B} = p_X^{-1} \mathcal{O}_S$) entails, for any $\mathcal{N} \in \text{D}_{\text{coh}}^{\text{b}}(\mathcal{D}_{X \times S/S})$, a natural morphism

$$R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{N}, \mathcal{M}) \rightarrow R\mathcal{H}om_{p_X^{-1} \mathcal{O}_S}(\text{DR } \mathbf{D} \mathcal{M}, \text{DR } \mathbf{D} \mathcal{N}).$$

When $\mathcal{N} = \mathcal{O}_{X \times S}$, we obtain a natural morphism

$$\text{DR } \mathcal{M} \rightarrow \mathbf{D}' \text{DR } \mathbf{D} \mathcal{M}, \quad \text{that is,} \quad {}^{\text{p}}\text{DR } \mathcal{M} \rightarrow \mathbf{D}^{\text{pDR}} \mathbf{D} \mathcal{M}.$$

Suppose now that $\mathcal{M} \in \text{D}_{\text{hol}}^{\text{b}}(\mathcal{D}_{X \times S/S})$. Recall that $\mathbf{D} \mathcal{M} \in \text{D}_{\text{hol}}^{\text{b}}(\mathcal{D}_{X \times S/S})$, so ${}^{\text{p}}\text{DR } \mathbf{D} \mathcal{M} \in \text{D}_{\mathbb{C}\text{-c}}^{\text{b}}(p_X^{-1} \mathcal{O}_S)$.

Hence, by biduality, we get a morphism

$$(6) \quad \mathbf{D}^{\text{pDR}} \mathcal{M} \leftarrow {}^{\text{p}}\text{DR } \mathbf{D} \mathcal{M}.$$

On the other hand, since $Li_{s_o}^*(\mathcal{M}) \in \text{D}_{\text{hol}}^{\text{b}}(\mathcal{D}_X)$ for each $s_o \in S$, the morphisms above induce isomorphisms

$$Li_{s_o}^*(\mathbf{D}^{\text{pDR}} \mathcal{M}) \simeq {}^{\text{p}}\text{DR } \mathbf{D} Li_{s_o}^*(\mathcal{M})$$

according to Proposition 2.1 and Proposition 3.1, where in the right hand side we consider the duality for holonomic \mathcal{D}_X -modules. Thus (6) is an isomorphism by Proposition 2.2 and the local duality theorem for holonomic \mathcal{D}_X -modules (see [13] and the references given there). q.e.d.

Example 3.12. Let X be the open unit disc in \mathbb{C} with coordinate x and let S be a connected open set of \mathbb{C} with coordinate s . Let $\varphi : S \rightarrow \mathbb{C}$ be a non constant holomorphic function on S and consider the holonomic $\mathcal{D}_{X \times S/S}$ -module $\mathcal{M} = \mathcal{D}_{X \times S/S} / \mathcal{D}_{X \times S/S} \cdot P$, with $P = x\partial_x - \varphi(s)$. It is easy to check that \mathcal{M} has no \mathcal{O}_S -torsion and admits the resolution $0 \rightarrow \mathcal{D}_{X \times S/S} \xrightarrow{P} \mathcal{D}_{X \times S/S} \rightarrow \mathcal{M} \rightarrow 0$, so that the dual module $\mathbf{D} \mathcal{M}$ has a similar presentation and is also \mathcal{O}_S -flat. The complex ${}^{\text{p}}\text{Sol } \mathcal{M}$ is represented by $0 \rightarrow \mathcal{O}_{X \times S} \xrightarrow{P} \mathcal{O}_{X \times S} \rightarrow 0$ (terms in degrees -1 and 0). Consider the stratification $X_1 = X \setminus \{0\}$ and $X_0 = \{0\}$ of X . Then $\mathcal{H}^{-1} {}^{\text{p}}\text{Sol } \mathcal{M}|_{X_1}$ is a locally constant sheaf of free $p_X^{-1} \mathcal{O}_S$ -modules generated by a local determination of $x^{\varphi(s)}$, and $\mathcal{H}^0 {}^{\text{p}}\text{Sol } \mathcal{M}|_{X_1} = 0$. On the other hand, $\mathcal{H}^{-1} {}^{\text{p}}\text{Sol } \mathcal{M}|_{X_0} = 0$ and $\mathcal{H}^0 {}^{\text{p}}\text{Sol } \mathcal{M}|_{X_0}$ is a skyscraper sheaf on $X_0 \times S$ supported on $\{s \in S \mid \varphi(s) \in \mathbb{Z}\}$.

For each x_0 we have

$$\begin{aligned} i_{x_0}^!({}^{\text{p}}\text{Sol } \mathcal{M}) &\simeq i_{\{x_0\} \times S}^{-1} R\mathcal{H}om_{\mathcal{D}_{X \times S}}(\mathcal{D}_{X \times S} \otimes_{\mathcal{D}_{X \times S/S}} \mathcal{M}, R\Gamma_{\{x_0\} \times S|X \times S} \mathcal{O}_{X \times S})[\dim X] \\ &\simeq i_{\{x_0\} \times S}^{-1} R\mathcal{H}om_{\mathcal{D}_{X \times S}}(\mathcal{D}_{X \times S} \otimes_{\mathcal{D}_{X \times S/S}} \mathcal{M}, B_{\{x_0\} \times S|X \times S}) \end{aligned}$$

where $B_{\{x_0\} \times S | X \times S} := \mathcal{H}_{[\{x_0\} \times S]}^1(\mathcal{O}_{X \times S})$ denotes the sheaf of holomorphic hyperfunctions (of finite order) along $x = x_0$ (cf. [16]). The second isomorphism follows from the fact that $\mathcal{D}_{X \times S} \otimes_{\mathcal{D}_{X \times S/S}} \mathcal{M}$ is regular specializable along the submanifold $x = x_0$ (cf. [7]).

Recall that the sheaves $B_{\{x_0\} \times S | X \times S}$ are flat over $p_X^{-1} \mathcal{O}_S$ because locally they are inductive limits of free $p_X^{-1} \mathcal{O}_S$ -modules of finite rank.

Since $i_{x_0}^!(\text{PSol } \mathcal{M})$ is quasi isomorphic to the complex

$$0 \rightarrow B_{\{x_0\} \times S | X \times S} |_{\{x_0\} \times S} \xrightarrow{P} B_{\{x_0\} \times S | X \times S} |_{\{x_0\} \times S} \rightarrow 0$$

it follows that the flat dimension over \mathcal{O}_S of $i_{x_0}^!(\text{PSol } \mathcal{M})$ in the sense of [4, §4] is ≤ 0 for any x_0 . Moreover, $\mathcal{H}^0 i_{x_0}^!(\text{PSol } \mathcal{M}) = 0$ and, if $x_0 \neq 0$, $\mathcal{H}^1 i_{x_0}^!(\text{PSol } \mathcal{M})$ is a locally free \mathcal{O}_S -module of rank 1. Hence the flat dimension of $i_{x_0}^!(\text{PSol } \mathcal{M})$ is ≤ 1 . This shows explicitly that $\text{PSol } \mathcal{M}$ satisfies the condition (Cosupp+) of Corollary 2.29.

4. APPLICATION TO MIXED TWISTOR \mathcal{D} -MODULES

Let $\mathcal{R}_{X \times \mathbb{C}}$ be the sheaf on $X \times \mathbb{C}$ of z -differential operators, locally generated by $\mathcal{O}_{X \times \mathbb{C}}$ and the z -vector fields $z \partial_{x_i}$ in local coordinates (x_1, \dots, x_n) on X . When restricted to $X \times \mathbb{C}^*$, the sheaf $\mathcal{R}_{X \times \mathbb{C}^*}$ is isomorphic to $\mathcal{D}_{X \times \mathbb{C}^* / \mathbb{C}^*}$.

A mixed twistor \mathcal{D} -module on X (see [12]) is a triple $\mathcal{T} = (\mathcal{M}', \mathcal{M}'', C)$, where $\mathcal{M}', \mathcal{M}''$ are holonomic $\mathcal{R}_{X \times \mathbb{C}}$ -modules and C is a certain pairing with values in distributions, that we will not need to make precise here. Such a triple is subject to various conditions. We say that a $\mathcal{D}_{X \times \mathbb{C}^* / \mathbb{C}^*}$ -module \mathcal{M} underlies a mixed twistor \mathcal{D} -module \mathcal{T} if \mathcal{M} is the restriction to $X \times \mathbb{C}^*$ of \mathcal{M}' or \mathcal{M}'' .

Theorem 1.1 is now a direct consequence of the following properties of mixed twistor \mathcal{D} -modules, since they imply that \mathcal{M} satisfies the assumptions of Theorem 1.2. If \mathcal{M} underlies a mixed twistor \mathcal{D} -module, then

- there exists a locally finite filtration $W_\bullet \mathcal{M}$ indexed by \mathbb{Z} by $\mathcal{R}_{X \times \mathbb{C}}$ -submodules such that each graded module underlies a pure polarizable twistor \mathcal{D} -module; then each $\text{gr}_\ell^W \mathcal{M}$ is strict and holonomic (see [14, Prop. 4.1.3] and [11, §17.1.1]), and thus so is \mathcal{M} ;
- the dual of \mathcal{M} as a $\mathcal{R}_{X \times \mathbb{C}^*}$ -module also underlies a mixed twistor \mathcal{D} -module, hence is also strict holonomic (see [12, Th. 12.9]); using the isomorphism $\mathcal{R}_{X \times \mathbb{C}^*} \simeq \mathcal{D}_{X \times \mathbb{C}^* / \mathbb{C}^*}$, we see that the dual $\mathbf{D} \mathcal{M}$ as a $\mathcal{D}_{X \times \mathbb{C}^* / \mathbb{C}^*}$ -module is strict and holonomic. q.e.d.

REFERENCES

- [1] M. Goresky and R.D. MacPherson, *Stratified Morse theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Folge 3 Band 14, Springer-Verlag, Berlin, Heidelberg, New York, 1988.
- [2] M. Kashiwara, *On the maximally overdetermined systems of differential equations*, Publ. RIMS, Kyoto Univ. **10** (1975), 563–579.
- [3] ———, *D-modules and microlocal calculus*, Translations of Mathematical Monographs, vol. 217, American Mathematical Society, Providence, RI, 2003.

- [4] ———, *t-Structure on the derived categories of \mathcal{D} -modules and \mathcal{O} -modules*, Moscow Math. J. **4** (2004), no. 4, 847–868.
- [5] M. Kashiwara and P. Schapira, *Sheaves on Manifolds*, Grundlehren der mathematischen Wissenschaften, vol. 292, Springer-Verlag, 1990.
- [6] ———, *Categories and sheaves*, Grundlehren der mathematischen Wissenschaften, vol. 332, Springer-Verlag, 2006.
- [7] Y. Laurent and T. Monteiro Fernandes, *Systèmes différentiels fuchsien le long d'une sous-variété*, Publ. RIMS, Kyoto Univ. **24** (1988), no. 3, 397–431.
- [8] Z. Mebkhout, *Le formalisme des six opérations de Grothendieck pour les \mathcal{D} -modules cohérents*, Travaux en cours, vol. 35, Hermann, Paris, 1989.
- [9] ———, *Le théorème de positivité, le théorème de comparaison et le théorème d'existence de Riemann*, Éléments de la théorie des systèmes différentiels géométriques, Séminaires & Congrès, vol. 8, Société Mathématique de France, Paris, 2004, pp. 165–310.
- [10] T. Mochizuki, *Asymptotic behaviour of tame harmonic bundles and an application to pure twistor D -modules*, vol. 185, Mem. Amer. Math. Soc., no. 869-870, American Mathematical Society, Providence, RI, 2007, arXiv: [math.DG/0312230](#) & [math.DG/0402122](#).
- [11] ———, *Wild harmonic bundles and wild pure twistor D -modules*, Astérisque, vol. 340, Société Mathématique de France, Paris, 2011.
- [12] ———, *Mixed twistor D -Module*, arXiv: [1104.3366](#), 2011.
- [13] L. Narváez Macarro, *The local duality theorem in \mathcal{D} -module theory*, Éléments de la théorie des systèmes différentiels géométriques, Séminaires & Congrès, vol. 8, Société Mathématique de France, Paris, 2004, pp. 59–87.
- [14] C. Sabbah, *Polarizable twistor \mathcal{D} -modules*, Astérisque, vol. 300, Société Mathématique de France, Paris, 2005.
- [15] ———, *Wild twistor \mathcal{D} -modules*, Algebraic Analysis and Around: In Honor of Professor M. Kashiwara's 60th Birthday (Kyoto, June 2007), Advanced Studies in Pure Math., vol. 54, Math. Soc. Japan, Tokyo, 2009, pp. 293–353, arXiv: [0803.0287](#).
- [16] M. Sato, T. Kawai, and M. Kashiwara, *Microfunctions and pseudo-differential equations*, Hyperfunctions and pseudo-differential equations (Katata, 1971), Lect. Notes in Math., vol. 287, Springer-Verlag, 1973, pp. 265–529.
- [17] P. Schapira, *Microdifferential systems in the complex domain*, Grundlehren der mathematischen Wissenschaften, vol. 269, Springer-Verlag, 1985.
- [18] P. Schapira and J.-P. Schneiders, *Index theorem for elliptic pairs*, Astérisque, vol. 224, Société Mathématique de France, Paris, 1994.
- [19] C. Simpson, *Higgs bundles and local systems*, Publ. Math. Inst. Hautes Études Sci. **75** (1992), 5–95.
- [20] ———, *Mixed twistor structures*, Prépublication Université de Toulouse & arXiv: [math.AG/9705006](#), 1997.

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